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Computation of the flow between a rotating and a stationary disk

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A numerical exploration of multiple-cell solutions for the flow between a rotating and a stationary disk is carried out systematically by the continuation method. The paper confirms and extends the work of Mellor, Chapple & Stokes. The results include the discovery of one-, two-, three- and five-cell solution regions which have not been reported before.

1. Introduction

The problem of the steady incompressible flow between two infinite rotating disks has attracted continued attention in the past because it offers the possibility of obtaining an exact solution to the Navier–Stokes equations. For, as Batchelor (1951) showed in his extension of von Kármán's (1921) solution to the problem of a single rotating disk, if the axial velocity of the flow is assumed to be independent of the radial distance, then the Navier–Stokes equations can be reduced to a two-point boundary-value problem for a system of two nonlinear ordinary differential equations. Lance & Rogers (1962) treated the problem numerically for various values of the ratio of the angular velocities of the two disks and for a range of solutions from Stokes flow to typical boundary-layer flows. Subsequently Mellor, Chapple & Stokes (1968) presented a rather complete numerical treatment of the problem when one disk is rotating while the other is stationary. In addition they have produced further analytical results and experimental data. They found multiple-cell solutions (a cell is defined by planes parallel to the disks on which the axial velocity vanishes). Mellor *et al.* discovered two one-cell branches, one of which – we call it the 'principal branch' – is the set of solutions discussed qualitatively by Batchelor (1951) and is the one for which the experimental data were taken, the other of which – we call it the 'von Kármán branch' – leads in the limit to von Kármán's (1921) solution for a single disk. Mellor *et al.* (1968) also found one two-cell and one three-cell solution branch.

Mellor *et al.* converted the Navier–Stokes partial differential equations into ordinary differential equations and treated the resulting two-point boundary-value problem as if it were an initial-value problem. They carried this out by selecting trial values for the missing initial conditions and integrating across the gap between the disks until a disk spacing was found at which the terminal boundary conditions were satisfied or until overflow occurred. Theirs appears to be a trial-and-error procedure with the disk spacing determined as a consequence of the solution.

In more recent work Ockendon (1972) discussed the asymptotic solution for steady-state flow above an infinite rotating disk with suction. Nguyen, Ribault & Florent (1975) presented numerical data on the multiple-cell solutions for flow between coaxial disks of Stewartson type (velocity entirely axial outside the boundary layer) and of Batchelor type, and also experimental data which indicate that the physical flow is of the Batchelor type.

In this paper we present through the continuation method of Roberts & Shipman (1967, 1968) an orderly way to explore for the multiple solutions associated with this problem by treating the two-point boundary-value problem as one in which the disk spacing is specified beforehand. The continuation method enables the principal branch of the one-cell solutions to be computed automatically and systematically starting from scratch, so to speak. With a minimum of manual intervention we were able to transfer to the von Kármán branch, and then by continuation to compute that branch of the one-cell solution. Further, by a manual exploration technique similar to that of Mellor *et al.*, we formed trial guesses for the missing initial conditions corresponding to multiple-cell solutions. Once these initial guesses had led to a converged solution (via a shooting method) we could generate automatically an entire branch of a multiple-cell solution by continuation. In this way we reproduced and extended the two- and three-cell branches of Mellor *et al.*

In the course of our work we have found one-, two-, three- and even five-cell solutions which have not been reported before.

While we have confined our computations to the problem in which only one disk is rotating in order to compare the results with those of Mellor *et al.*, our methods are equally applicable to the cases in which both disks are rotating, either in the same direction or in opposite directions.

2. Problem statement

Consider the physical situation of two disks, one stationary and one rotating, set a distance of $z = l$ units apart along the axis, where $z = 0$ refers to the stationary disk. For steady incompressible flow of a viscous fluid the governing equations consist of the equation of continuity and the three equations of motion. Under axial symmetry these equations are

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z} = 0, \tag{2.1a}$$

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u}{r}\right)+\frac{\partial^2 u}{\partial z^2}\right), \tag{2.1b}$$

$$\frac{u}{r}\frac{\partial}{\partial r}(rv)+w\frac{\partial v}{\partial z} = \nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{\partial^2 v}{\partial z^2}\right), \tag{2.1c}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right) \tag{2.1d}$$

and the boundary conditions are

$$\left.\begin{aligned} u(r,0) = v(r,0) = w(r,0) = 0, \\ u(r,l) = 0, \quad v(r,l) = \Omega r, \quad w(r,l) = 0, \end{aligned}\right\} \tag{2.2}$$

where u = radial velocity, v = tangential velocity, w = axial velocity, Ω = angular velocity of rotating disk, r = radial co-ordinate, z = axial co-ordinate $(0 \leqslant z \leqslant l)$, l = disk spacing, ν = kinematic viscosity and ρ = density.

Mellor *et al.* (1968), following Batchelor (1951), assumed as one form of the similarity solution the following:

$$\left.\begin{aligned} z = (\nu/\omega)^{\frac{1}{2}}\eta, \quad v = \omega r g(\eta), \quad w = -2(\omega\nu)^{\frac{1}{2}}h(\eta), \quad u = \omega r h'(\eta), \\ p/\rho = \omega\nu\hat{P}(\eta) + \tfrac{1}{2}\lambda\omega^2 r^2, \end{aligned}\right\} \tag{2.3}$$

which when substituted into (2.1) yield the nonlinear ordinary differential equations

$$h''' + 2hh'' - h'^2 = \lambda - g^2, \tag{2.4a}$$

$$g'' + 2hg' - 2h'g = 0, \tag{2.4b}$$

$$\hat{P}' = -2(h'' + 2hh') \tag{2.4c}$$

and which when substituted into (2.2) yield the boundary conditions

$$h(0) = h(\eta_l) = h'(0) = h'(\eta_l) = g(0) = 0, \quad g'(0) = 1, \tag{2.5}$$

where η_l is the final value of η, that is, the disk spacing. In addition to (2.3) and (2.4), Mellor *et al.* gave another scaling definition, which is particularly well suited to small values of the Reynolds number R, and an alternative boundary condition to the last condition in (2.5), namely $g(\eta_l) = 1$.

For our computations we converted the second- and third-order differential equations in (2.4) into a set of six first-order ordinary differential equations, which include the variable λ, by the definitions

$$y_1 = h, \quad y_2 = h', \quad y_3 = h'', \quad y_4 = g, \quad y_5 = g', \quad y_6 = \lambda. \tag{2.6}$$

The differential equations (2.4) now become

$$\left.\begin{aligned} \dot{y}_1 = y_2, \quad \dot{y}_2 = y_3, \\ \dot{y}_3 = -2y_1y_3 + y_2^2 + y_6 - y_4^2, \\ \dot{y}_4 = y_5, \quad \dot{y}_5 = -2y_1y_5 + 2y_2y_4, \quad \dot{y}_6 = 0 \end{aligned}\right\} \tag{2.7}$$

with the boundary conditions

$$y_1(0) = y_1(\eta_l) = y_2(0) = y_2(\eta_l) = y_4(0) = 0, \quad y_5(0) = 1. \tag{2.8}$$

Once (2.7) and (2.8) have been solved, the pressure $\hat{P}$ can be found from (2.4*c*) by quadrature. Although all our computations were carried out with the first-order system (2.7) and (2.8) since the existing computer program we applied to the problem is designed for such systems, we shall discuss our results in terms of the functions g and h of Mellor *et al.* (1968).

3. Solution method and continuation

We employed the shooting method to solve the two-point boundary-value problem consisting of the system of nonlinear ordinary differential equations (2.7) and the explicit boundary conditions (2.8). The shooting method is an iterative technique which is initiated with trial values of the missing 'initial conditions', in this case $h''(0)$ and $h'''(0)$. At each stage of the iteration 'corrected' values of the missing initial conditions are systematically computed, and under appropriate conditions converge to the true values. See Roberts & Shipman (1972) for a detailed discussion of the method.

The continuation method of Roberts & Shipman (1967, 1968) was devised to deal with unstable or numerically sensitive two-point boundary-value problems. These are problems in which, for trial values of the missing initial conditions close to but not equal to the true values, it is impossible to integrate the differential equations over the interval of interest, say $[t_0, t_f]$, without overflow at some intermediate t ($t_0 < t < t_f$). When applied to such cases, the continuation method begins by solving the original two-point boundary-value problem over the shorter interval $[t_0, t_1]$ ($t_0 < t_1 < t_f$) with the final values prescribed at t_f imposed at t_1. The converged values of the missing initial conditions are then taken as trial values for the missing initial conditions for the next two-point boundary-value problem, over the interval $[t_0, t_2]$ ($t_1 < t_2 < t_f$). Under favourable conditions, after a finite number n of steps, a point t_n will be reached such that $t_n = t_f$, at which time the original two-point boundary-value problem will have been solved. For each interval of the continuation method the boundary-value problem is solved by the shooting method. For more details on continuation see Roberts & Shipman (1967, 1968).

In the present application of continuation, while the mechanism is the same the spirit is somewhat different. Here we begin with the counterpart of t_1, that is a value of η_l, and solve (if possible) the two-point boundary-value problem (2.7) and (2.8) over the interval $[0, \eta_l]$. Then by continuation we solve the problem over the interval $[0, \eta_l + \Delta\eta_l]$, where $\Delta\eta_l$ may be positive or negative. Thus, if a solution is known for a certain value of η_l, other solutions may be found from it by continuation in a systematic fashion, and a complete branch of solutions generated.

4. Discussion of results

We began by computing the principal branch of the one-cell solution. To start the process we considered the limiting case in which $\eta_l = 0$ and the Reynolds number $R = 0$, where it can be argued that since there is no axial flow $h(\eta)$ is identically zero and therefore $h''(0) = h'''(0) = 0$. There is in fact an inconsistency in this limiting case: since there is no tangential flow on the stationary disk, but there is tangential flow on the moving disk, $g(0)$ is required to be both zero and non-zero. However, because our purpose in considering this case was to obtain initial guesses for the missing initial conditions $h''(0)$ and $h'''(0)$, and because we actually began the computation not with $\eta_l = 0$ but with a small non-zero value of η_l, this theoretical inconsistency was of no practical consequence. In fact we assigned the values $h''(0) = h'''(0) = 0$ as initial guesses for the missing initial conditions for the problem with $\eta_l = 1$. The shooting method converged in three iterations to the presumed true values of the missing initial conditions. Thus, starting with the converged initial conditions for $\eta_l = 1$, the principal branch of the single-cell solutions was computed by the continuation method, with $\Delta\eta_l$ first set equal to 0·5 up to $\eta_l = 2{\cdot}5$ and then set equal to 0·25 for the remainder of the computation. In table 1 (not shown)† we give some of the results of our computation method‡ interleaved with some of the results of Mellor *et al.* It can be seen that the two sets of results blend smoothly with one or two exceptions. These may be explained by the fact that when we checked the results of Mellor *et al.* by integrating forward with their initial conditions, first with a four-point Runge–Kutta method (with step size = 0·05) and then with the extrapolation method of Bulirsch & Stoer (1966), which adjusts the step size (set initially at 0·05) and the order of integration such that a preassigned accuracy is guaranteed, we found that the final conditions were met to an accuracy only of $|10^{-3}|$. On the other hand, with our converged values of the missing initial conditions, integration by either method generated terminal conditions that satisfied the final boundary conditions to an accuracy of $|10^{-11}|$ or better. This indicates that the initial values of Mellor *et al.* are slightly in error. As a check we recomputed the solution for $\eta_l = 2{\cdot}44$ and for $\eta_l = 6{\cdot}18$. We found that the missing initial conditions in our computations for $\eta_l = 2{\cdot}44$ are $h''(0) = -1{\cdot}000\,359\,450$ and $h'''(0) = 1{\cdot}701\,975\,910$ compared with the results $h''(0) = -1{\cdot}0$ and $h'''(0) = 1{\cdot}702$ of Mellor *et al.* For $\eta_l = 6{\cdot}18$ we obtained $h''(0) = -1{\cdot}218\,713\,168$ and $h'''(0) = 1{\cdot}409\,696\,064$, while Mellor *et al.* obtained $h''(0) = -1{\cdot}218\,715$ and $h'''(0) = 1{\cdot}40970$. Thus for these two cases differences in the initial conditions of the order of $|10^{-4}|$ are sufficient to account for the reported accuracy of the terminal conditions. Nevertheless, we confirmed the solution of Mellor *et al.* for the principal branch in all important respects, and furthermore were able to compute the solution automatically with no *a priori* information.

Since η_l decreases along the von Kármán branch of the one-cell solutions as given by Mellor *et al.*, we attempted to compute this branch by initially perturbing slightly the values of $h''(0)$ and $h'''(0)$ given in table 1 for $\eta_l = 9{\cdot}75$ in order to locate a point on this branch, after which we proceeded by continuation. To be explicit, we set as our trial values $h''(0) = -1{\cdot}218\,771\,879$ and $h'''(0) = 1{\cdot}4$ and by a trial-and-error process tried to solve the boundary-value problem for various values of η_l until the problem converged at $\eta_l = 6{\cdot}5$. This put us on the von Kármán branch, which we then followed by continuation until $\eta_l = 5{\cdot}25$. We could have continued further but, since we had achieved our aim of finding the von Kármán branch, we chose to explore other regions of interest. Table 2 (not

† Tables 1–5 are not exhibited in this paper, however copies of the complete tables are available from the authors or the JFM Editorial Office, DAMTP, Silver Street, Cambridge CB3 9EW.

‡ To facilitate comparison with the data of Mellor *et al.* (1968) in their table 2, we give in our tables 1–5 the same information, namely $h''(0)$, $h'''(0)$, η_l, $g(\eta_l)$, $g'(\eta_l)$, R, Λ and $G'(1)$.

shown) gives the results of applying the continuation method to the von Kármán branch with interleaved results of Mellor *et al.* As a possible alternative approach to locating a point on the von Kármán branch, we could have employed the asymptotic solution for $R \to \pm\infty$, the Bödewadt (1940) solution, as discussed by Mellor *et al.* (1968), to provide trial values for $h''(0)$ and $h'''(0)$ for some η_l.

We now consider the multi-cell cases. In the paper of Mellor *et al.* the impression is given that each one-cell, two-cell and three-cell region is isolated and unrelated to other multiple-cell regions. In our investigations we have found this to be only partially true. We learned that, once we could locate, say, a two-cell branch, then by continuity we could sometimes find a contiguous one-cell or three-cell region and we learned that the transition from one multiple-cell region to another could indeed be smooth. On the other hand we also discovered some regions which were in fact isolated.

Table 3 (not shown) lists the results of our exploration of a three-cell and adjacent two-cell regions. Interleaved with our data are the results of Mellor *et al.* With few exceptions the two sets of data blend well. As a starting point we chose for the trial initial conditions $h''(0) = -1{\cdot}0$ and $h'''(0) = 3{\cdot}48$ (which correspond to the converged conditions of Mellor *et al.* for $\eta_l = 5{\cdot}215$) and $\eta_l = 5{\cdot}25$. After these trial initial conditions converged to yield a three-cell solution we explored this branch for both positive and negative values of $\Delta\eta_l$ by continuation. For the range of initial conditions considered, the three-cell solutions lie in the interval $4{\cdot}74 \leqslant \eta_l \leqslant 5{\cdot}50$. We could not extend the solution by continuation below $\eta_l = 4{\cdot}74$. At $\eta_l = 4{\cdot}74$ our Reynolds number was -921,† which exceeds the limiting Reynolds number found by Mellor *et al.* for their three-cell data. Beyond $\eta_l = 5{\cdot}50$ there was a smooth transition from three- to two-cell solutions. This cell transition corresponds to $h''(0)$ passing from a negative value to zero at $\eta_l = 5{\cdot}510$ and then remaining positive for increasing values of η_l. The two-cell region in the interval $5{\cdot}10 \leqslant \eta_l \leqslant 6{\cdot}00$ appears to be a new discovery since it was not reported by Mellor *et al.* We were able to continue the two-cell branch mathematically from $\eta_l = 6{\cdot}00$ to $11{\cdot}7$, at which point we arbitrarily stopped our research. However, beyond $\eta_l = 6{\cdot}00$, Λ, the rescaled radial pressure gradient, became negative, and this implies that for some radial distance the pressure will become negative, which cannot occur in a real flow.

In the two-cell solutions the flow pattern is such that in the first cell the axial flow is towards the stationary disk and in the second cell the axial flow is away from the rotating disk. The two-cell solutions reported in table 3 differ from those reported by Mellor *et al.* since in table 3 $h''(0) > 0$, $\Lambda \approx 10^{-3}$ and $R \approx -400$, while in the data of Mellor *et al.* $h''(0) < 0$, $\Lambda \approx 2{\cdot}7$ and $R \approx -70$; see figure 1.

The data in table 3 were plotted as $h'''(0)$ *vs.* $-h''(0)$ in figure 1 with Reynolds numbers indicated at some of the points. The plot yielded a 'hook' at a Reynolds number of -921 similar to the 'hook' for the one-cell branch shown by Mellor *et al.* in their figure 2. They could not obtain a solution for the principal branch of the one-cell problem beyond a Reynolds number of 337.

Table 4 (not shown) lists the results of our exploration of adjacent two- and

† We have adopted the convention of Mellor *et al.* of reporting the Reynolds number as negative when $g(\eta)$ is negative.

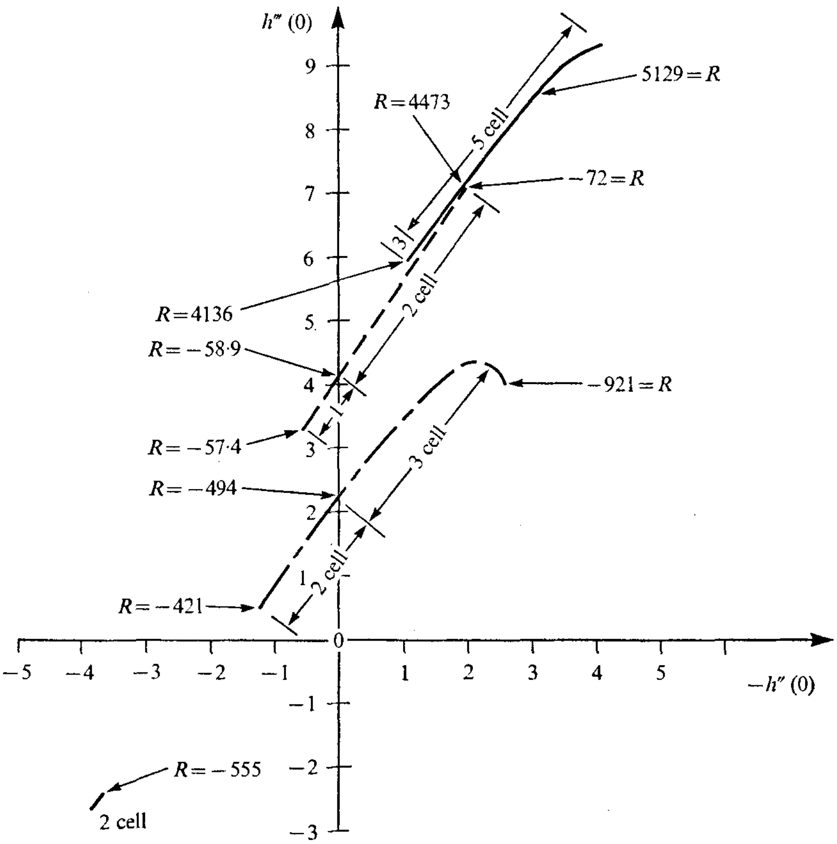

FIGURE 1. $h'''(0)$ *vs.* $-h''(0)$ for multi-cell flow regions. R = Reynolds number. – – –, table 3 data; —— — ——, table 4 data; ——, table 5 data.

one-cell regions. Interleaved with our data are the results of Mellor *et al.* The two sets of data blend well. As a starting point we chose for the trial initial conditions $h''(0) = -2{\cdot}0$ and $h'''(0) = 7{\cdot}235$ (which correspond to the converged conditions of Mellor *et al.* for $\eta_l = 6{\cdot}582$) and $\eta_l = 6{\cdot}6$. After these trial initial conditions had converged to yield a two-cell solution we explored this branch with both positive and negative values of $\Delta\eta_l$ by continuation. For the range of initial conditions considered the two-cell branch lies in the interval $6{\cdot}582 \leqslant \eta_l \leqslant 6{\cdot}945$. By a smooth transition the adjacent one-cell region in table 4 lies in the interval $7{\cdot}0 \leqslant \eta_l \leqslant 7{\cdot}2$. The table 4 data represent distinctly isolated solutions; see figure 1. For the range of initial conditions considered it is not possible to find solutions for $\eta_l < 6{\cdot}582$ or $\eta_l > 7{\cdot}2$. It is interesting to observe that, although both table 3 and table 4 report two-cell branches, the conditions under which two cells exist are quite different. For example, in table 3, $h''(0) > 0$, $\Lambda \approx 10^{-3}$, $R \approx -400$ and $G'(1) \approx -12$, while in table 4, $h''(0) < 0$, $\Lambda \approx 2{\cdot}7$, $R \approx -70$ and $G'(1) \approx -19$.

Although Mellor *et al.* felt that they had found all the one-cell solutions, our investigations have turned up another one-cell branch, as reported in table 4.

As $h''(0)$ changes sign from negative to zero the two-cell branch changes into a one-cell branch. Referring to figure 1, we have by continuation extended the two-cell branch across the $h'''(0)$ axis but were not able to extend it to the $h''(0)$ axis. As the $h'''(0)$ axis is crossed, the solution changes from having two cells to having one. The single-cell solution is characterized by flow away from the rotating disk as in figure 1 of Batchelor (1951), for the problem of a single rotating disk. If we compare the one-cell solution data in tables 1 and 2 with the one-cell solution data in table 4 we find that $h''(0)$ and $h'''(0)$ for the principal branch and the von Kármán branch are comparable while the initial conditions in table 4 are quite different. Furthermore, Λ is of the order of 10^{-1} for the principal branch, of the order of 10^{-2} for the von Kármán branch and of the order of 2·7 for the table 4 data. At $\eta_l \approx 7$, $R \approx 170$ for the principal branch, $R \approx -425$ for the von Kármán branch and $R \approx -58$ for the one-cell data of table 4.

We conjecture that any solution branch, with the exception of the principal one-cell branch, can be continued across the $h'''(0)$ axis, and that one cell will be lost in crossing this axis. A plausible argument for this conjecture runs as follows: in the neighbourhood of the stationary disk, that is, for $\eta \ll 1$, we have by a Taylor series expansion $h(\eta) \approx \frac{1}{2}h''(0)\,\eta^2 + \frac{1}{6}\lambda\eta^3$. If $h''(0) < 0$, then $h(\eta)$ will have a root, and consequently the flow will have a cell at (approximately)

$$\eta_0 = -3h''(0)/\lambda.$$

As $h''(0)$ increases to zero, η_0 also approaches zero, so that the corresponding cell disappears. When $h''(0)$ becomes positive η_0 becomes negative, so there is no cell corresponding to this value.

In the final phase of those of our computations described here, we sought multiple-cell solutions of higher multiplicity than those previously reported (that is, three-cell solutions). Our method of finding these high multiplicity solutions was to start with the known converged initial conditions for a two- or three-cell problem and integrate forward as in an initial-value problem over a larger interval than in the two- or three-cell problem until $\eta = 10$ or until overflow. If upon inspection of the profile we discovered that $h(\eta)$ changed sign more than three times, we took the η at the last sign change as the η_l for a two-point boundary value problem whose trial missing initial conditions were the $h''(0)$ and $h'''(0)$ of the two- or three-cell problem. By this preliminary exploration we were able to discover a five-cell branch. Table 5 (not shown) lists the results of our five-cell computations plus some adjacent two- and three-cell solutions. The data in table 5 were generated starting with $h''(0) = -2{\cdot}0$ and $h'''(0) = 7{\cdot}235$ (which correspond to a two-cell solution of Mellor *et al.* at $\eta_l = 6{\cdot}582$) and $\eta_l = 9{\cdot}6$ and continued for positive and negative values of $\Delta\eta_l$. Converged solutions for the five-cell branch were found for the interval $8{\cdot}66 \leqslant \eta_l \leqslant 9{\cdot}6$. A $\Delta\eta_l$ of $-0{\cdot}05$ was employed from $\eta_l = 9{\cdot}40$ to $\eta_l = 8{\cdot}80$. Overflow occurred when we attempted to continue the solution from $\eta_l = 8{\cdot}80$ to 8·75 with $\Delta\eta_l = -0{\cdot}05$. However, when we selected $\Delta\eta_l = -0{\cdot}02$ we were able to continue solving the problem successfully from $\eta_l = 8{\cdot}80$ to $\eta_l = 8{\cdot}66$. We did not attempt to pursue continuation

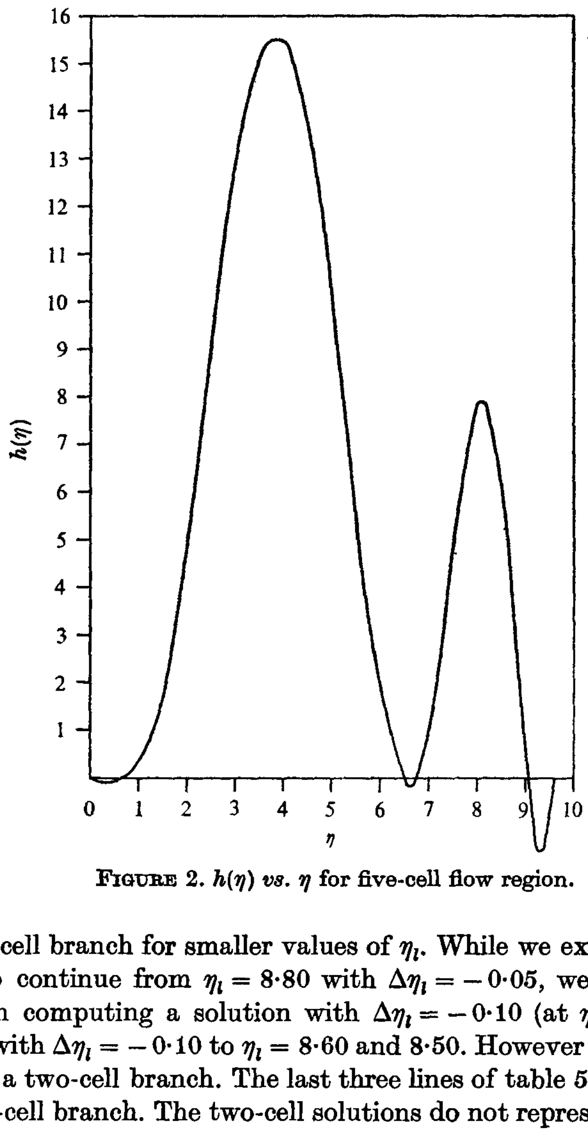

FIGURE 2. $h(\eta)$ *vs.* η for five-cell flow region.

for the five-cell branch for smaller values of η_l. While we experienced overflow in trying to continue from $\eta_l = 8{\cdot}80$ with $\Delta\eta_l = -0{\cdot}05$, we were nevertheless successful in computing a solution with $\Delta\eta_l = -0{\cdot}10$ (at $\eta_l = 8{\cdot}70$) and then continuing with $\Delta\eta_l = -0{\cdot}10$ to $\eta_l = 8{\cdot}60$ and $8{\cdot}50$. However the last three solutions are on a two-cell branch. The last three lines of table 5 represent the data for this two-cell branch. The two-cell solutions do not represent a smooth transition from the five-cell branch but rather a jump to a different quadrant in the $h''(0), h'''(0)$ plane; see figure 1. Since Λ is negative, we once again have a situation where the pressure will become negative at some radial distance. In the interval $9{\cdot}7 \leqslant \eta_l \leqslant 9{\cdot}8$ we have a three-cell branch, which was generated by a smooth transition from the five-cell branch starting at $\eta_l = 9{\cdot}6$ with $\Delta\eta_l = 0{\cdot}10$. We did not attempt to continue above $\eta_l = 9{\cdot}8$.

In figure 1 we may observe how the table 5 data for the five- and three-cell solutions blend with the two-cell data in table 4. This should not be too surprising since the data in table 5 were developed from one of the two-cell solutions in table 4.

A typical profile of $h(\eta)$ *vs.* η is given for the five-cell case at $\eta_l = 9{\cdot}6$ in figure 2. It will be noted that the third cell (counting from $\eta = 0$ or the stationary disk) is very narrow and that $h(\eta)$ is barely negative, so that one may wonder whether in fact five cells exist. To check this we took the initial conditions for the converged solutions of almost all the five-cell problems and recalculated the profiles as an initial-value problem by the precision integration method of Bulirsch & Stoer (1966). In every case we confirmed numerically that the five cells indeed existed and the boundary conditions were satisfied. For the five-cell problems Λ is of the order of 10^{-3} and the Reynolds number $R \approx 5000$.

As a practical consideration it is interesting to note that the torque exerted by the fluid on the rotating disk, which is proportional to $G'(1)$, increases with the number of cells generated. Therefore it takes more power to drive the rotating disk in a multi-cell state than in the single-cell state.

5. Conclusions

We have applied the continuation method to the computation of the flow between a rotating and a stationary disk. We have shown how the continuation method enables the principal branch of the single-cell solution to be generated automatically and systematically. The generation of the other branches of the solutions found by Mellor *et al.* (1968) could proceed automatically once a point on the branch had been found. In the course of our computations we confirmed and extended the results of Mellor *et al.* (1968), and in particular we have identified new one-, two- and three-cell branches, as well as a five-cell branch. From our investigations it would appear that there exists a multitude of solution branches. Determination of the distribution of the branches in, say, the $h''(0)$, $h'''(0)$ plane awaits further numerical or analytical study.

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